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An exact solution of the relativistic two-body problem with scalar interaction

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Abstract. We consider the problem of two classical point particles in circular orbits, each interacting with the retarded classical scalar field of the other. We find a unique solution for the massless field, but no solution for the massive field.

1. Introduction

Very few examples are known of exact solutions to the special relativistic two-body problem with physically reasonable retarded interactions, and with radiation effects included. In a previous paper [1] we have considered this problem with retarded classical electromagnetic interaction. We found a unique solution for the case when the two equal-mass, opposite-charge particles are diametrically opposed in circular orbits. The radiation reaction force was exactly compensated by the tangential component of the Lorentz force, and the system drew upon its infinite self-energy as the source of radiation. Here we wish to consider the same system, but with a retarded classical scalar interaction. We find again a unique solution for the case of a massless field, but no solution for the massive field case.

2. Method of solution

The classical scalar field of mass μ satisfies†

$$(\partial^2+\mu^2)\phi=-g\int\delta(x-z(s))\,\mathrm{d}s \tag{1}$$

where the right-hand side is the source due to a point particle with scalar charge g and worldline $z(s)$. We initially confine our attention to the case $\mu=0$. The retarded solution of (1) at the event x is then [2]

$$\phi(x)=-g/\rho \tag{2}$$

† We use a subscriptless notation for 4-vectors (see [2]). For example, we have $v=\mathrm{d}x/\mathrm{d}s$, $a=\mathrm{d}v/\mathrm{d}s$, $\partial=\partial/\partial x$ (with components $\partial/\partial x^\mu$), $\partial R=(1+a\cdot R)R/\rho-v$. The dot product is with respect to the metric $\eta=\mathrm{diag}(-1,1,1,1)$.

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where $\rho = -\boldsymbol{R} \cdot v$ is the invariant distance corresponding to the retarded separation in the rest frame. Here $R = x - z(s_0)$, where s_0 is the solution of $R \cdot R = 0$, $R^0 > 0$. The equation of motion of a particle of charge g and mass m in a field ϕ is [3, p 56]†

$$\frac{\mathrm{d}}{\mathrm{d}s}[(m + g\phi)v] = -g\partial\phi. \tag{3}$$

Thus if F_{ext} represents the external 4-force, we have

$$F_{\text{ext}} = ma = -g[\partial\phi \cdot (\eta - vv) - \phi a]. \tag{4}$$

However, like the Lorentz force law of electromagnetism, this equation does not include the interaction of the particle with its own field. Barut and Villaroel [4] have shown that when this is taken into account, the equation of motion becomes

$$ma = F_{\text{ext}} + \frac{g^2}{3}(\dot{a} - va^2) \equiv F_{\text{ext}} + F_{\text{rr}} \tag{5}$$

which is entirely analogous to the Lorentz-Dirac equation for electromagnetism, except for the factor $\frac{1}{3}$ instead of $\frac{2}{3}$ in the radiation reaction term F_{rr}. However, the nature of the external force F_{ext} is different and necessitates a reworking of our analysis of [1]. We wish to evaluate this force for two diametrically opposed particles in circular motion with separation $2r$, and to see whether the scalar Lorentz-Dirac equation (5) can be satisfied. To achieve this we calculate the space components of the external and radiation reaction forces. From equation (2), the field produced by the source particle (which we label 2) at the diametrically opposite point (the position of particle 1) is

$$\phi = -g_2[2r\gamma \cos\alpha(1 + v \sin\alpha)]^{-1} \qquad \nabla\phi = -g_2(\gamma \boldsymbol{v} - \gamma^{-2}\boldsymbol{r}/\rho)/\rho^2 \tag{6}$$

where $\gamma = (1 - v^2)^{-1/2}$. Here α is the retardation angle, the angle between the actual and retarded positions of the source particle, as observed from the other particle, which therefore satisfies the geometric retardation condition $v = \alpha/\cos\alpha$. Thus the space component of the scalar Lorentz-Dirac equation (4) for particle 1 is

$$m_1\gamma^2\boldsymbol{a} = -g_1[\boldsymbol{a}\gamma^2\phi + \nabla\phi] - \gamma^5 g_1^2 \boldsymbol{v}v^2/(3r^2). \tag{7}$$

Thus by taking the scalar product of (7) with unit vectors $\hat{\boldsymbol{r}}$ snd $\hat{\boldsymbol{v}}$ in the radial and tangential directions we have

$$-m_1\gamma^2 v^2/r = g_1\gamma^2 v^2/r - g_1\hat{\boldsymbol{r}} \cdot \nabla\phi \qquad 0 = g_1\hat{\boldsymbol{v}} \cdot \nabla\phi + \gamma^5 g_1^2 v^3/(3r^2). \tag{8}$$

If we now insert the field (6) and let $m_1 = m_2$ and $g_1 = g_2$ (like particles attract), we obtain the transcendental equation

$$1 - \tfrac{4}{3}\gamma^6 v^2[\cos\alpha(1 + v \sin\alpha)]^2 = 0 \tag{9}$$

for the retardation angle α, together with the retardation condition $v = \alpha/\cos\alpha$. Graphing the left-hand side of this equation, we see that it has a unique solution α_0 for v in $(0, 1)$. Using Brent's algorithm [5] for zero finding, we obtain numerically $\alpha_0 \approx 0.454\,64$. We note that this value is obtained from geometric considerations alone. We have therefore satisfied the scalar Lorentz-Dirac equation for particle 1, and so by symmetry, that for particle 2 also. To complete the solution it is merely necessary to solve the first equation (8) for r. Introducing the fundamental length $\lambda = g^2/m$, it gives $r/\lambda \approx 0.523\,65$.

† Note, however, that here we use units with $c = 1$.

3. The massive field

We wish to examine the possibility that a similar solution exists for the case of a massive scalar field. For this we make use of the retarded Green function for equation (1) [3, p 160]:

$$\begin{aligned} \Delta^{\text{ret}}(x) &= \frac{1}{2\pi}\,\delta(x^2) - \frac{\mu}{8\pi(-x^2)^{1/2}}\,J_1(\mu(-x^2)^{1/2}) \qquad & x^2 \leqslant 0 \\ &= 0 & \text{otherwise} \end{aligned} \tag{10}$$

where μ is the field mass and J_1 a Bessel function. For the case of our two diametrically opposed particles we have to evaluate the total field at say $z_1(0) = (0, r, 0, 0)$ due to the world line $z_2(t) = (t, -r\cos\omega t, r\sin\omega t, 0)$ of the source particle. The extra field at z_1 due to the mass μ will be given by the integral

$$\int \frac{J_1(\mu(-(z_1 - x)^2)^{1/2})}{(-(z_1 - x)^2)^{1/2}} \int_{-\infty}^{t_0} \delta(x - z_2(t))\,\mathrm{d}t\,\mathrm{d}^4x \tag{11}$$

where t_0 is the solution of $(z_1 - z_2(t))^2 = 0$. (Since we are concerned only with making this integral zero we disregard constant factors.) The spacetime interval between the events z_1 and z_2 is

$$[-(z_1 - z_2)^2]^{1/2} = [t^2 - 2r^2(1 + \cos\omega t)]^{1/2} \tag{12}$$

so that letting $\kappa = r/\lambda$ and $\tau = t/\lambda$, we have to check whether the integral

$$I(\mu) = \int_{-\infty}^{\tau_0} \frac{J_1(\mu\lambda f(\tau))}{f(\tau)}\,\mathrm{d}\tau \tag{13}$$

where $f(\tau) = \{\tau^2 - 2\kappa^2[1 + \cos(v\tau/\kappa)]\}^{1/2}$ is zero for some $\mu > 0$. (Here τ_0 is the solution of $f(\tau) = 0$.) If this were the case, then the net force from the massive field would be the same as in the massless case. However, it is easy to see that this cannot be the case, for a change of the variable of integration to $\mu\lambda f$ shows that the integral $I(\mu)$ is proportional to $1/\mu$, while a numerical integration shows $I(1)$ to be about unity for the values of r and v found previously. We therefore conclude that there does not exist a two-body solution of the type we are considering for the case of a massive field interaction.

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